

# Riemann-Liouville Fractional Einstein Field Equations

Joakim Munkhammar <sup>\*</sup>

March 26, 2010

## Abstract

In this paper we establish a fractional generalization of Einstein field equations based on the Riemann-Liouville fractional generalization of the ordinary differential operator  $\partial_\mu$ . We show some elementary properties and prove that the field equations correspond to the regular Einstein field equations for the fractional order  $\alpha = 1$ . In addition to this we show that the field theory is inherently non-local in this approach. We also derive the linear field equations and show that they are a generalized version of the time fractional diffusion-wave equation. We show that in the Newtonian limit a fractional version of Poisson's equation for gravity arises. Finally we conclude open problems such as the relation of the non-locality of this theory to quantum field theories and the possible relation to fractional mechanics.

## 1 Introduction

General relativity as a theory of gravitation is perhaps the most successful theory in physics to date [16]. Therefore many generalizations of general relativity have been created in order to possibly incorporate additional physics [10, 2]. The various generalizations includes for example adding extra dimensions and making the metric complex [10, 2, 16].

The modern formulation of fractional calculus was formulated by Riemann on the basis of Liouville's approach [11, 12, 15]. Despite its long history it has only recently been applied to physics where areas including

---

<sup>\*</sup>Studentstaden 23:230, 752 33 Uppsala, Sweden; joakim.munkhammar@gmail.com

fractional mechanics and fractional quantum mechanics have been established [7, 13, 14]. Certain approaches to fractional Friedmann equations, fractal differential geometry and fractional variational principles have recently been developed [1, 3, 14].

In this paper we shall utilize a fractional calculus operator to generalize the differential geometry approach to general relativity. Similar developments, in particular regarding fractional Friedmann equations, have been made with different kinds of fractional differentiation operators [6], however we shall utilize the simplest possible generalization using a special form of the Riemann-Liouville fractional derivative.

## 2 Theory

### 2.1 Fractional Calculus

Fractional calculus is the non-integer generalization of an  $N$ -fold integration or differentiation [11]. We shall use a special case of the Riemann-Liouville fractional differential operator as the generalization of the differential operator  $\partial_\mu$  used in general relativity. If  $f(x_\mu) \in C([a, b])$  and  $-\infty < x_\mu < \infty$  then:

$$\partial_\mu^\alpha f := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx_\mu} \int_{-\infty}^{x_\mu} \frac{f(t)}{(x_\mu - t)^\alpha} dt, \quad (1)$$

for  $\alpha \in ]0, 1[$  is here called the fractional derivative of order  $\alpha$  (It is a special case of the general Riemann-Liouville derivative, see [11, 12, 15] for more information). For  $\alpha = 1$  we define  $\partial_\mu^\alpha = \partial_\mu$  and we have the ordinary derivative.

### 2.2 Fractional Einstein field equations

We shall assume spacetime to be constituted by a metric of the same type as for ordinary Einstein field equations [16]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2)$$

with the usual condition for the covariant metric [10, 16]:

$$g_{\mu\nu} g^{\mu\rho} = \delta_\nu^\rho. \quad (3)$$

The metric can also be characterized via vielbein according to:

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (4)$$

The dynamics is governed by the Einstein field equations (here with fractional parameter  $\alpha$ ):

$$R_{\mu\nu}(\alpha) - \frac{1}{2}g_{\mu\nu}R(\alpha) = \frac{8\pi G}{c^4}T_{\mu\nu}(\alpha). \quad (5)$$

In this fractional approach we shall assume that the Riemann tensor is constructed from the metric on the basis of the Riemann-Liouville fractional derivatives (1). We have the Riemann tensor as follows:

$$R_{\mu\nu}(\alpha) = R_{\mu\lambda\nu}^\lambda(\alpha) = \partial_\lambda \Gamma_{\mu\nu}^\lambda(\alpha) - \partial_\nu \Gamma_{\mu\lambda}^\lambda(\alpha) + \Gamma_{\sigma\lambda}^\lambda(\alpha) \Gamma_{\mu\nu}^\sigma(\alpha) - \Gamma_{\sigma\nu}^\lambda(\alpha) \Gamma_{\lambda\mu}^\sigma(\alpha), \quad (6)$$

where we define the Christoffel symbol as:

$$\Gamma_{\nu\lambda}^\mu(\alpha) = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}), \quad (7)$$

and  $\partial_\mu^\alpha$  is the fractional derivative (1). We may also construct a covariant fractional derivative:

$$\nabla_\nu^\alpha A_\mu = \partial_\nu^\alpha A_\mu - A_\beta \Gamma_{\mu\nu}^\beta(\alpha), \quad (8)$$

for some four vector  $A_\mu$ . We also have the fractional geodesic equation:

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\beta\sigma}^\alpha(\gamma) \frac{dx^\beta}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \quad (9)$$

The correspondence to the traditional Einstein field equations is when  $\partial_\mu^\alpha$  is replaced by the traditional derivative  $\partial_\mu$  which is equivalent to  $\alpha = 1$  by definition. A very special feature of this field theory is that it is non-local since the fractional derivative (1) utilizes information from the metric at great distances to create the fractional derivative.

### 2.3 Linearized field equations

We can write the metric (2) as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (10)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and  $h_{\mu\nu}$  is higher order terms. To first order in  $h_{\mu\nu}$  we may neglect products of Christoffel symbols and thus arrive at [16]:

$$R_{\alpha\mu\beta\nu}(\gamma) = \partial_\beta^\gamma \Gamma_{\alpha\mu\nu}(\gamma) - \partial_\nu^\gamma \Gamma_{\alpha\mu\beta}(\gamma) \quad (11)$$

where the Christoffel symbol  $\Gamma_{\alpha\mu\nu}(\gamma)$  is fractional of order  $\gamma$  according (7) (here linearized):

$$\Gamma_{\alpha\mu\nu}(\gamma) = \frac{1}{2}(\partial_\nu^\gamma h_{\mu\alpha} + \partial_\mu^\gamma h_{\nu\alpha} - \partial_\alpha^\gamma h_{\mu\nu}) \quad (12)$$

This brings the the Riemann tensor to:

$$R_{\alpha\mu\beta\nu}(\gamma) = \frac{1}{2}(\partial_\mu^\gamma \partial_\beta^\gamma h_{\nu\alpha} + \partial_\alpha^\gamma \partial_\nu^\gamma h_{\mu\beta} - \partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu} - \partial_\mu^\gamma \partial_\nu^\gamma h_{\alpha\beta}). \quad (13)$$

We also have the Ricci tensor (in this approximation):

$$R_{\mu\nu}(\gamma) = \frac{1}{2}(\partial_\alpha^\gamma \partial_\mu^\gamma h_\nu^\alpha + \partial_\alpha^\gamma \partial_\nu^\gamma h_\mu^\alpha - \partial_\mu^\gamma \partial_\nu^\gamma h - \eta^{\alpha\beta} \partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu}) \quad (14)$$

where  $h \equiv h_\alpha^\alpha$ . We also have the Ricci scalar as:

$$R = \partial_\mu^\gamma \partial_\nu^\gamma h^{\mu\nu} - \eta^{\alpha\beta} \partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu}. \quad (15)$$

The field equations (5) now appears as:

$$\partial_\alpha^\gamma \partial_\mu^\gamma h_\nu^\alpha + \partial_\alpha^\gamma \partial_\nu^\gamma h_\mu^\alpha - \partial_\mu^\gamma \partial_\nu^\gamma h - \eta^{\alpha\beta} \partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu} - \eta_{\mu\nu}(\partial_\alpha^\gamma \partial_\beta^\gamma h^{\alpha\beta} - \eta^{\alpha\beta} \partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu}) = \frac{8\pi G}{c^4} T_{\mu\nu}(\gamma). \quad (16)$$

It is useful to introduce the following:

$$\bar{h} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad (17)$$

and if we now impose a form of fractional Lorenz gauge condition:

$$\partial_\beta^\gamma \bar{h}_\alpha^\beta = 0, \quad (18)$$

the linearized field equations (16) turns out to be:

$$\partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}(\gamma). \quad (19)$$

One might define these set of equations as the fractional gravitoelectromagnetic equations, which is a fractional analogue of the ordinary gravitoelectromagnetic equations in general relativity [9]. In the situation where  $T_{\mu\nu}$  vanishes we have:

$$\partial_\alpha^\gamma \partial_\beta^\gamma h_{\mu\nu} = 0, \quad (20)$$

which is a generalization of the time fractional diffusion-wave equation (see [8]). If we let  $\gamma = 1$  in (20) then we get:

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = 0, \quad (21)$$

which is the regular wave equation for gravitational waves in General Relativity [16].

## 2.4 Newtonian approximation

The field equations (5) can be written in a trace-reversed form as:

$$R_{\mu\nu}(\gamma) = \frac{8\pi G}{c^4} \left( T_{\mu\nu}(\gamma) - \frac{1}{2}T(\gamma)g_{\mu\nu} \right). \quad (22)$$

In order to find the Newtonian limit we let a test particle velocity be approximately zero:

$$\frac{dx^\beta}{d\tau} \approx \left( \frac{dt}{d\tau}, 0, 0, 0 \right), \quad (23)$$

and thus:

$$\frac{d}{dt} \frac{dt}{d\tau} \approx 0. \quad (24)$$

Furthermore we shall assume that the metric and its fractional derivatives are approximately static and that the square deviations from the Minkowski metric are negligible. This gives us the fractional geodesic equation as:

$$\frac{d^2x^i}{dt^2} \approx -\Gamma_{00}^i(\alpha), \quad (25)$$

and the fractional derivative of the Newtonian potential  $\phi$  is then:

$$\partial_i^\gamma \phi \approx \frac{1}{2}g^{i\alpha}(g_{\alpha 0,0} + g_{0\alpha,0} - g_{00,\alpha}) \approx -\frac{1}{2}g^{ij}g_{00,j} \approx \frac{1}{2}g_{00,i}. \quad (26)$$

We may also assume that:

$$g_{00} \approx -c^2 - 2\phi, \quad (27)$$

holds. We only need the 00-component of the trace-reversed field equations (22):

$$R_{00}(\gamma) = \frac{8\pi G}{c^4} \left( T_{00}(\gamma) - \frac{1}{2}T(\gamma)g_{00} \right). \quad (28)$$

The stress-energy tensor in a low-speed and static field approximation becomes:

$$T_{00}(\gamma) \approx \rho(\gamma)c^2, \quad (29)$$

where  $\rho(\alpha)$  is a form of fractional generalization of matter density  $\rho$  (also note that for  $\alpha = 1$  the ordinary matter density  $\rho$  is obtained). We thus get the 00-component of the Riemann tensor as:

$$\frac{8\pi G}{c^4} \left( T_{00}(\gamma) - \frac{1}{2} T(\gamma) g_{00} \right) \approx 4\pi G \rho(\gamma). \quad (30)$$

We also have:

$$\partial_i^\gamma \partial_i^\gamma \phi \approx \Gamma_{00,i}^i \approx R_{00}(\gamma) \approx 4\pi G \rho(\gamma), \quad (31)$$

which in the case  $\gamma = 1$  turns out to be the traditional Poisson's equation for gravity:

$$\partial_i \partial_i \phi = \nabla^2 \phi = 4\pi G \rho, \quad (32)$$

which is the expected result.

### 3 Conclusions

We have established a fractional generalization of Einstein field equations based on the Riemann-Liouville fractional generalization of the differential operator  $\partial_\mu$ . We showed some special properties including the correspondence to conventional Einstein field equations for  $\alpha = 1$ . We showed that the field equations by necessity are non-local due to the non-local nature of the Riemann-Liouville fractional derivative. In fact the field is non-local even for  $\alpha = 1 - \epsilon$  for some arbitrarily small  $\epsilon$ . In this situation the field equations are practically equivalent to the ordinary Einstein field equations, but they still have the non-local property. Furthermore we linearized the field equations and obtained a fractional wave equation with source, which is then the fractional equivalent of gravitoelectromagnetism in ordinary general relativity. We also investigated the Newtonian limit and arrived at a fractional generalization of the classical Poisson's equation for gravity. Many open questions remain, such as if the non-locality in some way may be connected to quantum field theories. Another issue is if there is any connection between this approach and the fractional stability of Friedmann equations in ordinary Einstein field equations [3]. Also further connections to fractional mechanics and other fractional and fractal approaches to gravity are open issues.

## References

- [1] D. Baleanu, S. Muslih, *On fractional variational principles*, Adv. in Frac. Calc. Part. 2, p.115-126, (2007).
- [2] A. Einstein, A Generalization of the Relativistic Theory of Gravitation, The Annals of Mathematics, Second Series, Vol. 46, No. 4, pp. 578-584, (1945).
- [3] A. El-Nabulsi, *Fractional Unstable Euclidean Universe*, EJTP 8 p.111 (2005).
- [4] A. El-Nabulsi, *Fractional Nottale's Scale Relativity and emergence of complexified gravity*, Chaos, Solitons and Fractals 42 p.2924-2933 (2009).
- [5] A. El-Nabulsi, *Cosmology with a fractional action principle*, Rom. Rep. in Phys. Vol. 59, No. 3, p. 763-771, (2007).
- [6] V.V.Kobelev, *The variant of post-Newtonian mechanics with generalized fractional derivatives*, CHAOS 16, 043117 (2006).
- [7] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E 66, 056108 (2002).
- [8] F. Mainardi, *The time fractional diffusion-wave equation*, Radiophys. and Quant. Electr., Vol. 38, No. 1-2 (1995).
- [9] B. Mashhoon, *Gravitoelectromagnetism: A Brief Review*. *arXiv:gr-qc/0311030v2* 17 Apr 2008.
- [10] J. Munkhammar, *Linearization of Moffat's Symmetric Complex Metric Gravity*, *arXiv:0909.3586v1 [gr-qc]* (2009).
- [11] J. Munkhammar, *Riemann-Liouville fractional derivatives and the Taylor-Riemann series*, UUDM project report 2004:7 (2004).
- [12] J. Munkhammar, *Fractional Calculus and the Taylor-Riemann series*, RHIT U. J. Math. 6(1) (2005).
- [13] L. Oparnica, *Generalized fractional calculus with applications in mechanics*, Matematicki Vesnik 53 p.151-158 (2001).
- [14] E. Rabei, B. Ababneh, *Hamilton-Jacobi fractional mechanics*, J. Math. Anal. Appl. 344 p.799-805 (2008).

- [15] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Amsterdam, (1993).
- [16] G. Øyvind, H. Sigbjørn *Einstein's General Theory of Relativity*. New York: Springer, (2007).